PHILOSOPHY OF LOGIC AND LANGUAGE

WEEK 5: MODEL-THEORETIC CONSEQUENCE

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OVERVIEW

Last week, I discussed various strands of thought about the concept of **LOGICAL CONSEQUENCE**, introducing Tarski's **MODEL-THEORETIC** account.

This is the view that a conclusion is a logical consequence of a set of premises IFF there is no **MODEL** in which the premises are all true and the conclusion is false.

This week, we'll look in a bit of detail at two problems for this account, the problem of **LOGICAL CONSTANTS** and some influential objections raised by John Etchemendy.

LOGICAL CONSTANTS

Last week, we wondered about the difference between the following two arguments:

ARGUMENT 1

1. Everyone smokes and everyone drinks
2. So, everyone smokes and drinks

ARGUMENT 4

1. John is a bachelor
2. So, John is not married

On formal accounts, such as Tarski's, this is explained in terms of a difference between the logical forms of the two expressions.

Roughly, the logical form of an argument is what we obtain by replacing its non-logical expressions with schematic letters.

The idea is then that, in the case of **ARGUMENT 4**, there is a way of replacing these schematic letters, or assigning them meanings, such that the result is not truth-preserving.

By contrast, in the case of **ARGUMENT 1**, there is *not* a way of replacing the schematic letters, or assigning them meanings, such that the result is not truth-preserving.

This assumes that the words 'bachelor' and 'married' are non-logical, with the result that the logical form of **ARGUMENT 4** is:

1. **a** is a **F**
2. So, **a** is not **G**

But why not assume instead that they are *logical* expressions, with the result that the logical form of **ARGUMENT 4** is:

1. **a** is a bachelor
2. So, **a** is not married?

This is the problem of **LOGICAL CONSTANTS**: how are logical expressions or constants to be distinguished from non-logical ones?

Early on, Tarski seemed to have held that there was no principled distinction to be drawn, and that the choice of logical constants was largely pragmatic.

Later on, in work with Steven Givant, he took a more optimistic view. I'll sketch this view today, and look at an alternative solution next week.

PERMUTATION INVARIANCE

The central thought behind Tarski's later work is that logical expressions do not **DISCRIMINATE** between different objects or individual.

This is a version of the idea, mentioned briefly last week, that logic is **TOPIC-NEUTRAL**, applying to any subject matter whatsoever.

More precisely, Tarski's idea is that the logical expressions are those that are *invariant* under arbitrary *permutations* of the domain of objects.

A **PERMUTATION** of a domain D of objects is a one-to-one mapping from D onto D.

For example, suppose our domain is the set of 21st century US presidents, {George W. Bush, Barack Obama, Donald Trump}. The permutations of the domain include:

PERMUTATION 1

George W. Bush → Barack Obama
Barack Obama → Donald Trump
Donald Trump → George W. Bush

PERMUTATION 2

George W. Bush → Donald Trump
Barack Obama → Barack Obama
Donald Trump → George W. Bush

Given the notion of a permutation, we can introduce the notion of **INVARIANCE** under a permutation of a domain.

First, an *object* or *individual* O in the domain is invariant under a permutation of that domain IFF the object to which that permutation maps O is O itself.

Thus, none of the individuals in our domain is invariant under **PERMUTATION 1**, though Barack Obama is invariant under **PERMUTATION 2**.

Second, a *set* S of objects in the domain is invariant under a permutation of that domain IFF the set of objects to which that permutation maps the members of S is S itself

So the set of 21st century Republican presidents, {George W. Bush, Donald Trump}, is invariant under **PERMUTATION 2**, but not **PERMUTATION 1**.

Third, an ordered *n-tuple* T of objects in the domain is invariant under a permutation of that domain IFF the ordered n-tuple of objects to which that permutation maps the members of T is T itself.

So the ordered pair <Barack Obama, Barack Obama> is invariant under **PERMUTATION 2**, but not **PERMUTATION 1**.

This gives us a handle on a sense in which the sorts of entities that serve as the **EXTENSIONS** of expressions in a domain may be invariant under permutations of that domain.

We can then say that an expression is **LOGICAL** IFF its extension in each domain (meaning what it does) is invariant under all permutations of that domain.

To see how this works, consider the name 'John'. Its extension in any given domain is an object — which generally *won't* be invariant under permutations of the domain.

Similarly, the extension of the predicate 'is a bachelor' in any given domain is a *subset* of the domain, and also generally won't be invariant under permutations of the domain.

By contrast, the extension of the predicate 'is an object' in any given domain is the domain itself, which *is* invariant under permutations of the domain.

Similarly, the extension of the predicate 'is not an object' in any given domain is the empty set, which is *also* invariant under permutations of the domain.

So the name 'John' and predicate 'is a bachelor' come out as non-logical, while the predicates 'is an object' and 'is not an object' come out as logical.

What about connectives and quantifiers? We can think of their extensions as functions from n-tuples of sets of variable assignment to sets of variable assignments.

The extension of 'and' in a domain, for example, will be the function that maps each *pair* of sets S_1 and S_2 of variable assignments over that domain to their intersection, $S_1 \cap S_2$.

And the extension of 'some object' in a domain will be the function that maps each set of variable assignments S over that domain to the set of variable assignments that differ at most in x from *some* variable assignment in S.

Each of these functions is also invariant under permutations of the domain. The extensions of 'and' and 'some object' are thus also invariant under permutations of the domain.

PROBLEMS

Permutation Invariance is not without its problems, however. I'll mention just two of them.

PROBLEM 1: if no two objects have exactly the same mass, the extension of 'has exactly the same mass as' in a domain will be the same as 'is identical to'.

Moreover, this extension is invariant under permutations of the domain. So both expressions turn out to be logical.

But should the distinction between logical and nonlogical expressions turn on matters of contingent fact, such as whether any two objects have exactly the same mass?

It is tempting to try to fix this by appealing to metaphysically or even conceptually possible domains. But that won't help fix ...

PROBLEM 2: the extension of the predicate 'is a married bachelor' in any given domain is the empty set.

But as we have already seen, the empty set is always invariant under permutations of the domain. So 'is a married bachelor' comes out as logical!

ETCHEMENDY'S OBJECTIONS

John Etchemendy famously offers two objections designed to show that the model-theoretic account of logical consequence is theoretically inadequate.

CONCEPTUAL ADEQUACY

The first objection is that the model-theoretic account of logical consequence is **CONCEPTUALLY** inadequate.

On the model-theoretic account, remember, an argument is logically valid IFF there are no models in which its premises are true and its conclusion is false.

According to Etchemendy, this leaves something essential out of account: the logical validity of an argument provides a **GUARANTEE** that the argument is truth-preserving.

It perhaps *follows* from the fact that an argument is logically valid that there are no models in which its premises are true and its conclusion is false.

(Though Etchemendy in fact disputes this: this is the **UNDERGENERATION** problem, which I will mention briefly below.)

But its logical validity does not *consist* in there being no models in which its premises are true and its conclusion is false.

According to Etchemendy, the model-theoretic account of logical consequence thus makes a mistake akin to that of mistaking the symptoms of a disease for the disease itself.

In order to defend the model-theoretic account of logical consequence, we might try any of the following three strategies.

FIRST, we could try to deny that the logical validity of an argument provides the sort of guarantee that Etchemendy claims it does.

Etchemendy seems to think that logical validity provides some sort of conceptual or *a priori* warrant for the belief that the argument is truth-preserving.

In other words: if an argument is logically valid, and one understands the premises and conclusion, then one is in a position to know that the conclusion is true if the premises are true.

But while this is plausible in the case of many logically valid arguments, it is not obviously true in every case. (Think of long, complicated proofs.)

SECOND, we could try to argue that the model-theoretic account captures the guarantee in question.

For example, suppose that we grant that it is part of the concept of logical validity that a logically valid argument is truth-preserving in all possible worlds.

We might then try to argue that the model-theoretic account captures this, on the grounds that claims about the existence of models are modal claims. See, e.g., Gila Sher (1996).

THIRD, we could accept that logical validity provides some sort of guarantee, and that the model-theoretic account doesn't capture this, but deny that it matters.

On this view, the point of the model-theoretic account is not to give a **CONCEPTUAL ANALYSIS** of the concept of logical consequence.

Rather, it is to provide a theoretically useful refinement of a certain pre-theoretic notion. (Compare: the difference between the concepts of *recursive* and *computable* functions.)

EXTENSIONAL ADEQUACY

Etchemendy's second objection is that the model-theoretic account of logical consequence is **EXTENSIONALLY** inadequate.

He thinks the model-theoretic account both **OVERGENERATES**, i.e. declares as logically valid arguments that are *not* logically valid ...

... and that it **UNDERGENERATES**, i.e. declares as logically *in*valid arguments that are not logically invalid.

Etchemendy's focus, however, is on *over*generation. But he does not think that the model-theoretic account overgenerates in first-order logic.

Thanks to an argument from George Kreisel (1967), known as the **SQUEEZING ARGUMENT**, it can be shown that the model-theoretic account does *not* overgenerate in first-order logic.

In order to find examples of arguments which are truth-preserving in all models but not logically valid, Etchemendy therefore focuses on second-order logic.

The argument turns on the **CONTINUUM HYPOTHESIS**. This is the hypothesis that there is no set whose cardinality is between that of the integers and the real numbers.

It is possible to use nothing but logical expressions of second-order logic to formulate a sentence which is true in all second-order models IFF the continuum hypothesis is true.

Call this sentence **S**. Its negation, ¬**S**, is true in all second-order models IFF the continuum hypothesis is false.

Now consider the following arguments:

ARGUMENT 1

1. Donald Trump is a Republican
2. So, **S**

ARGUMENT 2

1. Donald Trump is a Republican
2. So, **¬S**

If the continuum hypothesis is true, then **S** is true in all models, and **ARGUMENT 1** is declared logically valid.

If the continuum hypothesis is false, then **¬S** is true in all models, and **ARGUMENT 2** is declared logically valid.

So either way, one of **ARGUMENT 1** and **ARGUMENT 2** is declared logically valid. But, Etchemendy claims, neither of them is in fact logically valid.

Why not? The thought *seems* to be that they can only be logically valid if either the continuum hypothesis or its negation is a logical truth.

But it is not the case that either the continuum hypothesis or its negation is a logical truth.

SUMMARY

We've seen that formal accounts of logical consequence generally, and Tarski's model-theoretic account in particular, have to face the problem of **LOGICAL CONSTANTS**.

This is the problem of distinguishing logical expressions or constants from non-logical ones.

Early on, Tarski seems to have taken a pragmatic attitude to this problem, but later on, opted for an account of the distinction in terms of **PERMUTATION INVARIANCE**.

We saw some problems with this account. And we'll look at an alternative solution next week.

We've also looked at Etchemendy's objections to the model-theoretic account of logical consequence.

The first objection is that the account is **CONCEPTUALLY** inadequate, mistaking the symptoms of logical consequence for logical consequence itself.

The second objection is that the account is **EXTENSIONALLY** inadequate, and in particular that it overgenerates.

Etchemendy's argument for this focuses on the case of second-order logic, and an example involving the continuum hypothesis.