PHILOSOPHY OF LOGIC AND LANGUAGE

WEEK 2: FREGE'S LOGICAL INNOVATIONS

JONNY MCINTOSH

OVERVIEW

Last week:

- the conception of logic underpinning Frege's **LOGICISM**
- in particular, maximal generality and normativity

This week:

- Frege's innovations in logic
- the philosophical ideas that led to them

I will focus on the views in Frege's first book, *Begriffsschrift*.

INTRODUCTION

What does Frege need a logic *for*? The short answer: to establish logicism.

To do that, Frege needs a system in which he can present "gap-free" proofs, i.e. proofs in which every premise and rule of inference is made explicit.

By inspecting proofs of the fundamental laws of arithmetic, we can then determine whether they rely on anything other than logical generalities and rules of inference.

But Frege quickly found that natural language was inadequate for the task, and so invented his Begriffsschrift, or concept-script, in its place.

BEFORE FREGE

LOGICAL TRADITIONS

Logic before Frege comprised two main traditions, Aristotelian and Stoic, which had been brought together to some extent by George Boole (1815-1864).

The heart of the Aristotelian tradition was the theory of the **SYLLOGISM**. It distinguished four basic forms of sentences:

- A: All Fs are Gs
- E: No Fs are Gs
- I: Some Fs are Gs
- O: Some Fs are not Gs

The theory of the syllogism specifies how valid arguments can be constructed out of sentences of these forms.

For example, *Barbara*:

1. All humans are animals
2. All animals are mortal
3. So, all humans are mortal

The theory of the syllogism is a theory of **TERMS**. The expressions that replace the schematic letters 'F' and 'G' are count nouns, like 'human' and 'mortal'.

It can be (somewhat artificially) extended to handle arguments involving proper names, instead of count nouns. But it doesn't cover arguments like:

1. It is raining
2. If it is raining then the streets are wet
3. So the streets are wet

You learnt to handle these when you learnt **PROPOSITIONAL** logic, which was the logic developed by a rival tradition, that of the Stoics.

Where the Aristotelian theory of the syllogism is a theory of terms, this is a logic of propositions or **SENTENCES**.

Boole brought the two traditions together in a system of **ALGEBRAIC LOGIC**.

He effects this bringing together by treating the propositions studied by the Stoics as being of a form of the sort of propositions studied by Aristotelians.

For example, 'If it is raining then the streets are wet' becomes 'All times at which it is true to say it is raining are times at which it is true to say the streets are wet'.

THREE PROBLEMS

The logic that emerged out of this tradition wasn't adequate for Frege's purposes. The problems are nicely illustrated by three arguments.

FIRST, the following two-step argument (Rumfitt 1994):

1. Every integer is divisible by itself
2. So seven is divisible by seven
3. So some integer is divisible by seven

Intuitively, both the move from 1 to 2 *and* the move from 2 to 3 are valid. Why?

The argument moves from an ascription of one property, that of being divisible by itself, to every number to an ascription of another property, that of being divisible by seven, to seven.

The Stoic tradition, with its focus on sentences, had nothing to offer in accounting for this.

Crucially, however, neither did the Aristotelian tradition. For implicit in the theory of the syllogism is the assumption that each sentence has *one* subject and *one* predicate.

We can think of the predicate in the sentence 'seven is divisible by seven' as being *either* 'is divisible by itself', explaining the move from 1 to 2...

or as being 'is divisible by seven', explaining the move from 2 to 3. But we cannot regard it as being *both*.

SECOND, the following simple argument (Dudman 1976):

1. Everyone is rich or everyone is famous
2. Everyone is rich or famous

Stoic logic was designed to handle sentences like 1; Aristotelian logic sentences like 2. Neither could handle both.

Boole's logic offered means for formalising both kinds of sentences, but not in such a way that the word 'or' occurs in both with the same interpretation.

But unless we can formalise both sentences in such a way that the word 'or' occurs in both with the same interpretation, we can't explain the validity of the inference.

THIRD, consider the following argument:

1. Every student is clever
2. Every relative of a student is a relative of someone clever

To account for the validity of this number, we need a logic that can handle sentences of **MULTIPLE GENERALITY**, such as 'Every mouse fears some cat'.

Frege's logic can account for the validity of all of these arguments. I'll explain the main innovations, and the philosophical ideas that led to them.

These are perhaps best thought of as revolving around two distinctions, between *judgements* and their *contents*, and between *functions* and their *arguments*.

JUDGEMENT & CONTENT

JUDGEMENT FIRST

Frege's first move is to reverse the traditional order of explanation, which explains judgements in terms of concepts, and instead explain concepts in terms of judgements.

To see how this helps address the problems with the traditional logic, consider the judgement that seven is divisible by seven.

On the traditional view, this judgement is arrived at by subsuming something under a concept. But which concept, exactly? *Being divisible by seven*? *Being divisible by itself*?

There seems to be no principled reason for preferring one over the other, and if we do, we run into difficulty into accounting for the validity of the two-step argument.

Frege instead starts with the judgement and, as we will see, allows both ways of decomposing its content into concepts to have equal standing.

The first step is to explain the notion of the **CONCEPTUAL CONTENT** of a judgement in terms of the notions of judgement and inference.

Inference, in Frege's view, is a special form of judgement. Suppose I make the inference: John smokes and drinks, so John smokes.

Here, I don't simply make one judgement (that John smokes and drinks) and then another (that John smokes). Rather, I make the second judgement *on the basis of* the first.

Not just any case in which one makes one judgement on the basis of another counts as an inference for Frege, however.

Consider: it is raining, therefore the streets are wet. Here I make the judgement that the streets are wet on the basis of the judgement that it is raining.

But in so doing, I reason in a way that is not licensed by the laws of logic — my reasoning does not conform to the norms that the laws of logic determine.

Only those judgements made on the basis of other judgements in a way that conforms to the laws of logic count as inferences for Frege.

Thus armed with the notions of judgement and inference, Frege says that two judgements have the same conceptual content IFF they are **INFERENTIALLY EQUIVALENT**, i.e. ...

any inference that can be made on the basis of the one, given a set of further judgements, can also be made on the basis of the other, given the same set of further judgements.

"the contents of two judgements can differ in two ways: either the conclusions that can be drawn from one when combined with certain others also always follow from the second when combined with the same judgements, or else this is not the case. The two propositions 'At Plataea the Greeks defeated the Persians' and 'At Plataea the Persians were defeated by the Greeks' differ in the first way. Even if a slight difference in sense can be discerned, the agreement predominates. Now I call that part of the content that is the *same* in both the *conceptual content*." (Frege *Begriffsschrift* §3)

Frege goes on to identify the conceptual content of a *sentence* as being that of the judgement expressed by the assertion made by means of that sentence.

AN APPLICATION

An important application of Frege's criterion: on the basis of the judgements that Hesperus is Phosphorus, and that Phosphorus is a planet, one can judge that Hesperus is a planet...

But one *cannot* judge that Hesperus is a planet on the basis of the judgements that Hesperus is Hesperus, and that Phosphorus is a planet.

It therefore follows that the content of the judgement that Hesperus is Hesperus is not the same as the content of the judgement that Hesperus is Phosphorus.

Frege goes on to identify the conceptual content of a *sentence* as being that of the judgement expressed by the assertion made by means of that sentence.

PROBLEMS

The notion of inference cannot be understood in terms of necessary truth preservation, on pain of identifying the contents of all judgements whose truth-values necessarily coincide.

Neither can it be understood in terms of proof-theoretic deducibility, on pain of making the notion of conceptual content dependent upon the language in which judgements are expressed.

Frege came to hold that inferences cannot be based upon false judgements. If so, his criterion gives the absurd result that all false judgements have the same conceptual content.

It is impossible to judge that it is both raining and not raining, or that I am not me. But these are presumably not identical conceptual contents.

Frege doesn't employ this criterion of identity for conceptual contents in later work, perhaps for these sorts of reasons.

THE FREGE POINT

Whether or not Frege was right to think that content can be explained in terms of judgement, it is widely thought that he was right to draw a sharp distinction between the two.

Consider the fact that a sentence can be uttered to express a judgement and also *without* expressing a judgement, e.g. when it occurs as the antecedent of a conditional.

For example, in uttering the sentence 'If it is raining then the streets are wet', I utter the sentence 'It is raining' *without* expressing the judgement that it is raining.

But it seems that the sentence must have the same content there as it does when I use it to express the judgement that it is raining.

For otherwise, it seems impossible to account for the validity of reasoning that employs *modus ponens*.

This is the so-called **FREGE POINT**, and forms the basis of Peter Geach's argument against noncognitivism, the view that sentences such as 'Murder is wrong' are not truth-apt.

The distinction is enforced in Frege's logic by the distinction between what he calls **THE CONTENT STROKE**, '–', and **THE JUDGEMENT STROKE**, '|'.

The content stroke is used on its own, as in '–A', to express a content. To express a judgement with a given content, we add the judgement stroke, as in '⊢A'.

Having drawn the distinction, Frege argues that the distinction between *affirmative* and *negative* judgements is properly understood as a distinction between contents.

Frege goes on to introduce a notation and semantics for negative and conditional contents — equivalents of the connectives '¬' and '→'.

Also: axioms governing these and a rule of inference (*modus ponens*). Together with an implicit rule of substitution, these form a complete axiomatisation of propositional logic.

FUNCTION & ARGUMENT

This is all very impressive, but Frege's greatest innovation was his treatment of quantification. We start with judgements and their contents...

...as expressed by sentences. Consider, for example, the sentence 'seven is divisible by seven' again.

The traditional approach distinguished within this sentence a unique subject, the name 'seven', and a unique predicate, 'is divisible by seven'.

Frege abandons this approach altogether, and instead focuses on the idea that the sentence can be understood to be an instance of various different patterns.

For example:

- 'Seven is divisible by seven'
- 'Fourteen is divisible by seven'
- 'Nine is divisible by seven'

Here, we consider the pattern instantiated by all those sentences obtained by replacing the first occurrence of the name 'seven' and replacing it with another.

We can think of the invariant pattern as a **FUNCTION**, '[ ] is divisible by seven', which yields a sentence as **VALUE** given a name as an **ARGUMENT**.

"If, in an expression (whose content need not be a judgeable content), a simple or a complex symbol occurs in one or more places, and we think of it as replaceable at all or some of its occurrences by another symbol (but everywhere by the same symbol), then we call the part of the expression that on this occasion appears invariant the function, and the replaceable part its argument." (Frege *Begriffsschrift* §9)

Complex expressions quite generally, not just sentences, can be regarded in this way. And one and the same expression can be broken up into function and argument in various ways.

For example:

- 'Seven is divisible by seven'
- 'Seven is divisible by six'
- 'Seven is divisible by one'

Here the function is 'Seven is divisible by [ ]'.

For another example:

- 'Seven is divisible by seven'
- 'Nine is divisible by nine'
- 'Thirteen is divisible by thirteen'

Here the function is '[ ] is divisible by [ ]'.

Yet another example:

- 'Seven is divisible by seven'
- 'Nine is divisible by three'
- 'Thirteen is divisible by twenty'

Here the function is '[ ] is divisible by ( )', which takes *two* arguments.

Why is this significant? By exploiting this way of breaking sentences down into functions and their arguments, Frege is able to provide a revolutionary account of generality.

GENERALITY

According to Frege, a general sentence of the form 'Everything is F' is to be understood, not as the value of the function of the form '[ ] is F' given the argument 'everything'...

... but rather as stating that the function of the form '[ ] is F' yields a truth (or, as Frege has it, "is a fact") for every argument.

In predicate logic, we express this with the formula, $\forall x Fx$. Frege's notation is quite different, but the idea is essentially the same.

But the idea really comes into its own in its handling of **MULTIPLE GENERALITY**, and sentences such as 'Everything is divisible by everything'.

Frege's key thought here is that these are to be thought of as constructed in stages. Consider 'Everything is divisible by everything'.

We start with a singular sentence, such as 'Seven is divisible by seven'.

We think of it as the value of the function 'Seven is divisible by [ ]', given the argument 'seven'.

We can then form the general sentence, 'Seven is divisible by everything', stating that this function yields a truth for every argument.

We then think of this general sentence as the value of the function '( ) is divisible by everything', given the argument 'seven'.

And finally form the doubly general sentence, 'Everything is divisible by everything', stating that *this* function yields a truth for every argument.

The resulting account of generality allows us to solve all three problems.

Accounting for the validity of the two-step argument:

1. Every integer is divisible by itself
2. So seven is divisible by seven
3. So some integer is divisible by seven

And for the validity of the simple argument:

1. Everyone is rich or everyone is famous
2. Everyone is rich or famous

And, of course, multiple generality:

1. Every student is clever
2. Every relative of a student is a relative of someone clever

This enables Frege to establish that a core component of arithmetical reasoning, mathematical induction, is based entirely on the laws of logic, making a start on his logicist program.

SUMMARY

We've looked at the main problems with the logical tradition Frege inherited, and seen how he overcame them.

Two distinctions played a key role. First, the distinction between *judgements* and their *contents*.

Second, the distinction between *functions* and their *arguments*. This latter plays a key role in Frege's revolutionary account of generality.

Next week, we'll look at how Frege's views developed in the early 1890s, particularly his splitting of the notion of content into two parts, **SENSE** and **REFERENCE**.